

## SCES SUMMARY TALK

**Theory Perspective: SCES 2016.**Piers Coleman<sup>a,b</sup><sup>a</sup>Center for Materials Theory, Department of Physics and Astronomy, Rutgers University, 136 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA<sup>b</sup> Department of Physics, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK.**ARTICLE HISTORY**

Compiled August 22, 2016

**ABSTRACT**

New discoveries and developments in almost every area of correlated electron physics were presented at SCES 2016. Here, I provide a personal perspective on some of these developments, highlighting some new ideas in computational physics, discussing the “hidden order” challenges of cuprate and heavy electron superconductors, the mysterious bulk excitations of the topological Kondo insulator  $\text{SmB}_6$  and new progress in research on quantum spin ice, iron based superconductors and quantum criticality.

**KEYWORDS**

Correlated electrons; quantum matter; topological kondo insulators; spin liquids; quantum criticality

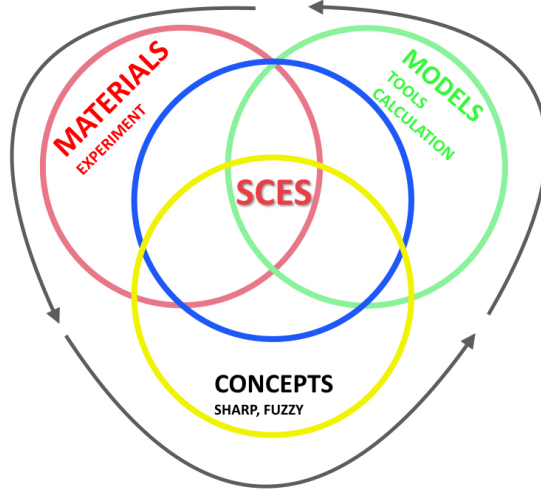
**1. Introduction: A Deluge of discovery and mystery.**

One of the great insights of the 20th century physics is that matter can acquire wholly unexpected new properties we call “*emergence*”, from the collective behavior of interacting quantum particles that lie within the material. The quest to discover, understand and harness materials with such novel *emergent* properties is the 21st century frontier of “*Strongly Correlated Electron Systems*” (SCES). In May 2016, the International Conference on Strongly Correlated Electron Systems, SCES 2016 convened for the first time in China, at Zhejiang University in historic Hangzhou.

SCES is a field of research that continues to surprise its most optimistic participants, with a deluge of discovery and new mysteries that confront us each year. Here I’d like to share with you some of the exciting developments that caught my eye at this meeting, complimenting Joe Thompson’s experimentally focused summary by emphasizing a theoretical perspective.

First a few words about the forces that drive discovery in our field. Theoretical research into strongly correlated materials lies at the intersection of three important areas: materials, concepts and models (see Fig. 1). Here are some of the topics that you might have heard about at this meeting:

- Materials and Experiment: heavy electron, organic, cuprate, ruthenate, iron-



**Figure 1.** The convective cycle of discovery in Strongly Correlated Electron Physics.

based, irridates and pyrocholores; laser ARPES, ultra-high magnetic fields, dHvA, scanning tunneling microscopy, NMR, RIXS, Raman and neutron spectroscopy to name a few.

- Concepts: emergence, order parameter, topological order, quasiparticle, Dirac, Majorana and Weyl Fermions, spin liquids and spin ice, pseudo-gap, strange metals and quantum criticality, Motttness, Many Body Localization, Skyrmion lattices and topological order.
- Models and methods: Heisenberg, Ising, Anderson, Hubbard, Kondo and Kitaev models. Renormalization, Landau Ginzburg theory, Path integrals, approximate  $1/N$ ,  $\epsilon$  and  $1/D$  expansions, Slave bosons and rotors, ab-initio density functional theory, dynamical mean field theory, Monte Carlo methods, density matrix renormalization and tensor network methods.

One of the most powerful drivers of discovery is the convective cycle between experiment and theory:

$$\begin{aligned}
 \text{new materials/experiment} &\Rightarrow \text{wild ideas} \Rightarrow \text{concepts} \Rightarrow \text{models} \Rightarrow \text{predictions} \\
 &\Rightarrow \text{new materials/experiment} \dots
 \end{aligned}
 \tag{1}$$

Theorists often feel most secure working “top-down” from well-established models and *ab-initio* methods, a vital part of the discovery process. Yet gems of discovery are to be found by those willing to move outside their comfort zone, seeking new ideas and insights in material discovery and experiments. To prospect amidst the real-world complexity of materials may sometimes seem daunting, but it is here that the rewarding task of building models and launching new interpretations often takes place. Here, the key tools are more phenomenological, such as the Landau theory and scaling approaches. The “bottom-up” approach theoretical condensed matter physics is valuable to experimentalists for it leverages their new data and occasionally provides the key to new concepts and the next generation of microscopic theories.

Another aspect of discovery are interdisciplinary ideas, borrowed from one another branch of physics or another class of material and boldly applied in a new context. Such an approach can seed amazing breakthroughs. For example, it is our fields' adoption of Feynman diagram and field theory methods born in Quantum Electrodynamics that drove the early revolution in Many Body Physics[1], including the BCS theory of superconductivity; aspirations towards similar breakthroughs continue with efforts to use holographic methods borrowed from string theory[2]. But even within condensed matter physics, who would have imagined that Kondo insulators and the Quantum Hall effect would eventually become linked?

$$\begin{aligned} \text{Quantum Hall Effect} &\rightarrow \text{Topological Matter} \rightarrow \text{Topological Insulators} \\ &\rightarrow \text{Topological Kondo insulators,} \end{aligned} \quad (2)$$

or that the discovery of cuprate superconductors would inspire a plethora of new magnetic and superconducting materials?

$$\text{cuprate superconductors} \rightarrow \begin{cases} \text{ruthenates} \\ 115 \text{ heavy fermion superconductors} , \\ \text{irridates} \end{cases} \quad (3)$$

or that the extension of the concept of vortex lattices in superconductors into the magnetic domain would lead to skyrmion lattices, a new development with serious applications? Each of these are important examples of the power of analogy and interdisciplinary collaboration that we need to strongly encourage in SCES.

## 2. New Computational Methods

In the 1960s, a combination of Monte-Carlo methods with Onsager's exact solution to the Ising model, paved the way for a grand revolution in understanding in statistical mechanics and critical phenomena[3]. One of our dreams today is to forge a similar breakthrough in our study and simulation of interacting quantum systems. At this meeting, two new developments caught my eye.

- *New progress on the fermion sign Problem of Quantum Monte Carlo.* In quantum systems, the Monte-Carlo method is a way of computing thermodynamic partition function by writing it as a Feynman sum over configurations weighted by the action of each configuration:

$$Z_Q = \text{Tr}[e^{-\beta H}] = \sum_{\{M\}} e^{-S[M]}. \quad (4)$$

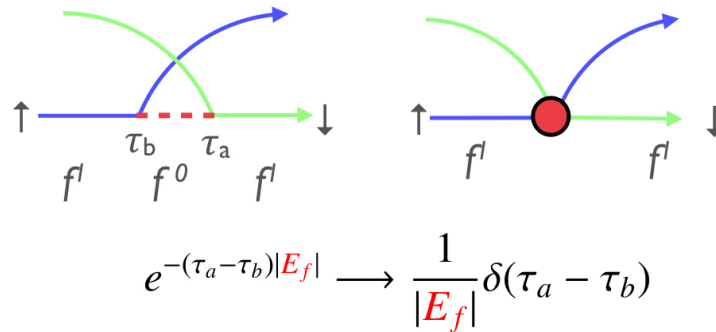
where the weight factor,  $e^{-S[M]} = W_{\uparrow}[M]W_{\downarrow}[M]$  is determined by the partition function  $W_{\sigma}[M]$  ( $\sigma = \uparrow, \downarrow$ ) of fermions moving in the field of the order parameter  $\{M\}$ . When fermions exchange, the wavefunction changes sign a minus sign so the quantum mechanical weight associated with a given configuration is often negative. Monte Carlo is only designed to deal with positive probabilities, and if negative weight configurations proliferate, Monte Carlo method breaks down. This is the "fermion sign problem".

At this meeting, Li, Jiang and Yao[4, 5] injected some new optimism into this problem, pointing out that the fermion sign problem is really basis dependent, which motivated them to search for new ways to formulate the path integral to mitigate the Fermion sign problem. One well known situation where sign problems disappear is the attractive Hubbard model, for which  $W_{\uparrow}[M]W_{\downarrow}[M] > 0$ . In this case, the Hubbard-Stratonovich Hamiltonian describing the fermion motion has time-reversal symmetry. This guarantees that  $W_{\downarrow}[M] = W_{\uparrow}[M]^*$ , so the resulting weight factor  $e^{-S[M]} = |W_{\uparrow}[M]|^2 > 0$  is positive. (Unfortunately, this is not true for the repulsive  $U$  Hubbard model, because the Hubbard Stratonovich transformed Hamiltonian breaks time-reversal symmetry.) Yao et al. [4, 5] have found that by splitting the fermion field into its real Majorana components,  $c = (\gamma_1 + i\gamma_2)$ , one can classify and identify new classes of “Majorana Time-reversal Symmetries” which guarantee that the sign problem is absent. I think this is a very interesting new direction, and I suspect that we have more to learn by examining the topological symmetries of models written in the Majorana language.

- One area where there is scope for progress, is in the implementation of Wilson’s renormalization concept into numerical methods. With the development of density matrix and tensor-network approaches, this is rapidly becoming a boom area of computational condensed matter physics. One advance that caught my eye was in a simple implementation of renormalization in the dynamical mean field approximation (DMFT). DMFT treats quantum many body problems as an impurity, or cluster immersed in a self-consistently determined bath. At the heart of this approach, is an impurity solver which computes electron self-energies. One popular approach here is to evaluate impurity electron self-energies by continuous-time Monte-Carlo methods. The random sampling of impurity histories necessarily involves a lot of virtual high energy processes that can, in principle be eliminated through the use of the renormalization group.

At SCES2016, Changming Yue, Yillan Wang and Xi Dai[6] showed how an efficient continuous time solver for Dynamical mean theory can be accomplished by integrating out high energy virtual charge fluctuations, replacing them by an effective spin scattering amplitude. For example, in an Anderson model, the virtual charge fluctuations

$$f^1 \rightleftharpoons f^0 \rightleftharpoons f^1 \quad (5)$$



**Figure 2.** Replacement of a high frequency virtual charge fluctuation by a single vertex enables the DMFT approach to incorporate the key features of the Schrieffer-Wolff transformation.

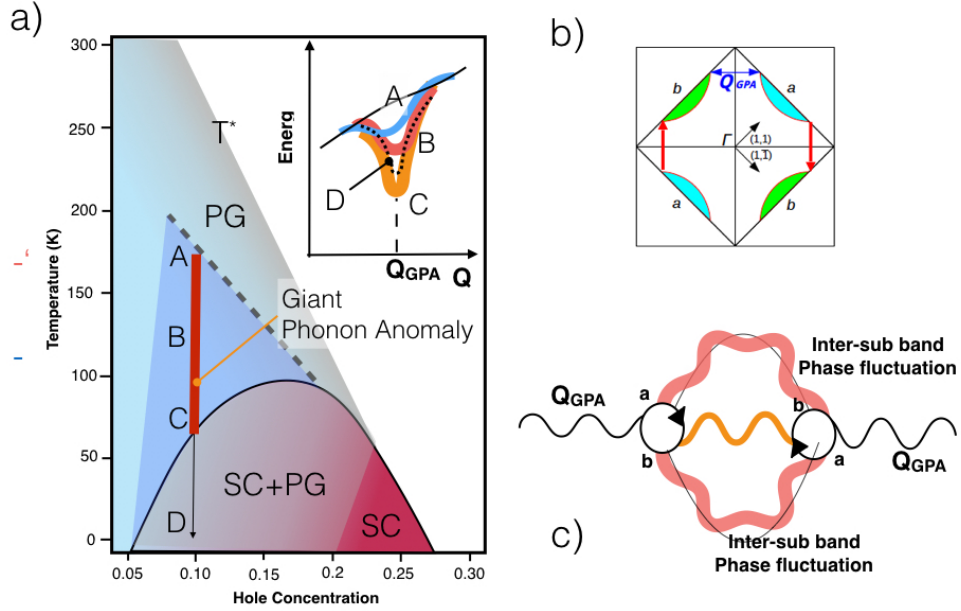
occurring at times  $\tau_b$  and  $\tau_a$  is associated with an amplitude  $e^{-(\tau_a-\tau_b)|E_f|}$  where  $|E_f|$  is the ionization energy of the atom. When integrated over time, this amplitude becomes  $1/|E_f|$ , allowing the replacement of the two virtual fluctuations by an effective spin exchange vertex (Fig. 2). This replacement is of course, the well-known “Schrieffer Wolff transformation” [7] that folds the Anderson model into the Kondo model. I was delighted to see that this same piece of physics has been made to accelerate a DMFT solver, so that in effect, rather than asking the computer to carry out the Schrieffer Wolff transformation millions of times a second, the renormalization is included at the outset. Yue et al demonstrated how their approach could be used to efficiently model the low temperature ARPES spectral function of the Kondo lattice system CeCoIn<sub>5</sub> [6] down to much lower temperatures. This appears to be an important practical development and perhaps first step along the road towards a fuller integration of the DMFT with the Wilsonian renormalization group.

### 3. Competing and Hidden Order

A continual challenge to SCES, is the discovery and identification of new forms of quantum order. Certain forms of order, such as pair or multipolar density waves are very difficult to detect directly, and may only be visible through their thermodynamic or indirect influence on other excitations, giving rise to “hidden order” (HO). The under-doped cuprates and URu<sub>2</sub>Si<sub>2</sub> both exhibit such hidden order. In the cuprate superconductors the development of a pseudo-gap [8] in the density of states in the under-doped region of the phase diagram is widely thought to be associated with one or more forms of hidden order (Fig. 3). In URu<sub>2</sub>Si<sub>2</sub>, a large moment antiferromagnet develops below 18K at 1.5GPa, but when the pressure is released, the antiferromagnetic order is replaced by hidden order, indicated by a large specific heat anomaly, with a Fermi surface geometry characteristic of a staggered order parameter, yet without antiferromagnetism (Fig. 4) [9]. Whereas pseudo-gap order in the underdoped cuprates tends to suppress superconductivity, hidden order in URu<sub>2</sub>Si<sub>2</sub> actually induces it. In both cases, the nature of the hidden order and its relationship with superconductivity are a long-standing, and unresolved mysteries.

At this meeting two new results on this topic caught my eye.

- **Giant Phonon Anomaly.** Maurice Rice discussed the under-doped cuprates, bringing to light an unusual *Giant Phonon Anomaly* (GPA) seen to develop within the pseudo-gap phase [11]. Below a certain temperature, considerably below the pseudo-gap temperature ( $T^*$ ), resonant X-ray scattering measurements show that phonons at particular wavevectors  $Q_{GPA}$  shift their energies and become damped. The onset of the Giant Phonon Anomaly is seen to coincide with the observation of a Josephson plasmon mode, associated with the development of interlayer pairing fluctuations [13]. The wavevector  $Q_{GPA}$  at which the anomaly develops matches with the momentum required to scatter between the Fermi arcs of the underdoped normal state (Fig. 3a,b). Remarkably, the phonon broadening (but not the frequency shift) disappears when the system goes superconducting. This has led Rice and collaborators [12] to propose a theory in which the phonon damping is driven by their coupling to an overdamped Leggett mode involving relative phase fluctuations between the developing d-wave condensates formed in the (x,y) and (x,-y) directions (Fig. 3c).

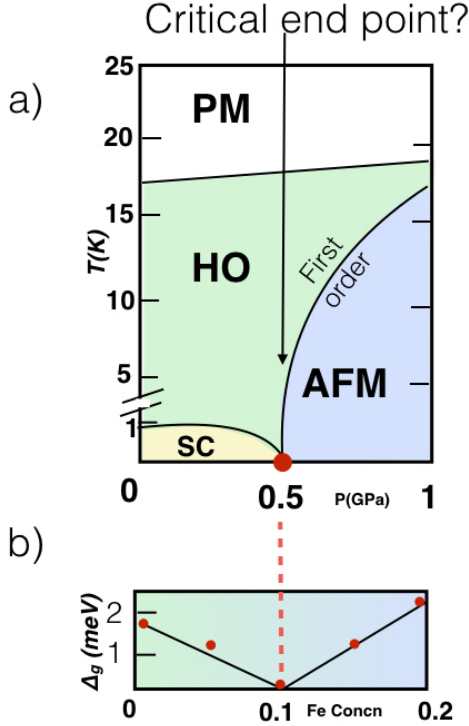


**Figure 3.** (a) Phase diagram of under-doped cuprates, sketched from [10], showing the pseudogap phase delineated by  $T^*$ . The Giant Phonon Anomaly (GPA) develops at a lower temperature, characterized by the growth of superconducting fluctuations. Inset: phonons at wavevector  $Q_{GPA}$  soften and broaden at onset A, sketched from [11]. In the superconducting phase, the signal remains soft but sharpens. (b) The wavevector  $Q_{GPA}$  links the Fermi arcs causing inter-sub band transitions. (c) Phase fluctuations (Leggett mode) couple to inter-sub band fluctuations to produce the Giant Phonon anomaly (sketched from [12])

- **Hidden order in  $URu_2Si_2$ .** Girsh Blumberg's group used Raman scattering to shed new light on a possible relationship between Hidden Order to magnetism in  $URu_2Si_2$ . In both the low pressure HO phase and the high pressure AFM phase, there is a collective Ising mode that can be probed by both neutron and Raman Scattering. By using iron doping to tune from the HO to the AFM phase, they observe that the Ising mode closes at the transition point between the two phases. This intriguing result suggests that

- (1) the transition between the HO and the AFM at absolute zero is continuous, forming a critical end point at the bottom of the first order line[16] that separates the two phases at finite temperatures and
- (2) at the transition, there is a gapless excitation that links the the HO and the AFM phase, suggest that they can be continuously transformed into one-another. If true, this has many important implications - for example, it appears to suggest that the HO phase breaks time-reversal symmetry.

These new developments pose many fascinating challenges to the theoretical community. I am not, for example aware of any microscopic model for the interaction of Raman modes with the magnetic and crystal field states of an f-electron atom. Another aspect that intrigues me, is whether the superconducting fluctuations in the pseudogap might also play a role in the strange metal phase that develops at higher doping.



**Figure 4.** (a) The pressure-temperature phase diagram of  $\text{URu}_2\text{Si}_2$ , showing how the release of pressure causes antiferromagnetism to transform into a state of Hidden order with superconductivity. (b) Sketch of the excitation gap seen by Kung et al [14, 15] using Raman scattering (using in iron-doping instead of pressure), showing how the excitation gap collapses at the zero temperature transition between Hidden order and AFM.

Rice argued that high-momentum pair fluctuations (“pair density wave”) fluctuations play an important role in the development of Fermi arcs. *Could do high-momentum pair fluctuations persist outside the pseudogap region into the strange metal phase of the cuprates?* Such pair fluctuations tend to develop phase coherence between electrons and holes moving in the same directions. Could they, I wonder, be connected with the mysterious observation of two transport relaxation rates, a longitudinal current relaxation rate  $\Gamma_{tr} \sim T$  which is linear, and a Hall current relaxation rate  $\Gamma_H \sim T^2$  which is quadratically dependent on temperature [17] ?

#### 4. $\text{SmB}_6$ and Strange and Topological Insulators

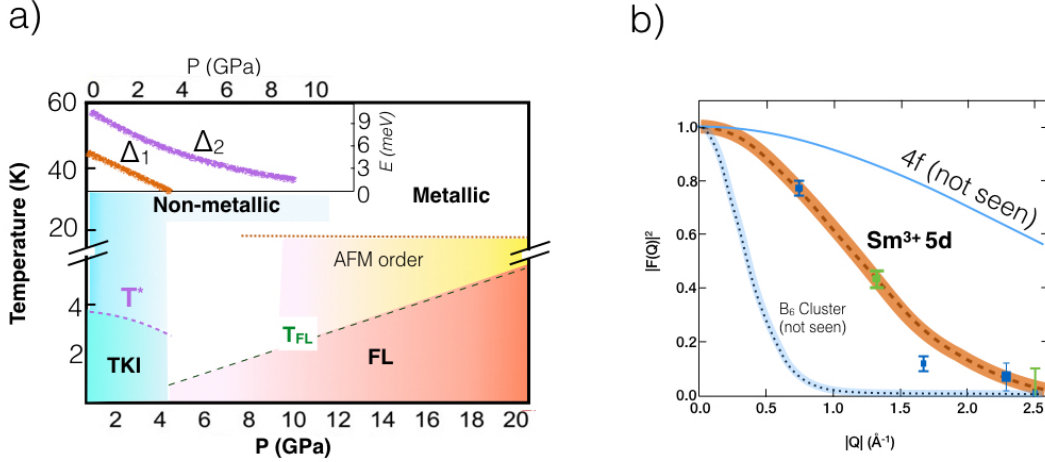
It is a tribute to the resiliency of the science of correlated quantum materials, that a narrow gap insulator,  $\text{SmB}_6$  discovered almost 60 years ago, is still generating new insights into quantum matter.

Over the past five years, it has become evident that the strong spin-orbit coupling in heavy fermion materials has the capacity to drive topological phase transitions. One way in which this can take place, is through the band-crossing of odd parity f-states and even parity d-states in Kondo insulators, giving rise to a topological Kondo insulator[18]. In f-electron systems, there is the additional possibility that interactions might drive qualitatively new forms of topological quantum matter. At this conference  $\text{SmB}_6$  seemed to offer just such a possibility.

Here, I’d like to mention three separate developments:

- The group of Liling Sun[19] presented a new set of high-pressure measurements that case new light into the phase diagram of  $\text{SmB}_6$ (Fig. 5a). This new data confirms that the topological conductivity plateau surfaces up to 4GPa of pressure,





**Figure 5.** (a) Sketch of pressure phase diagram of Kondo insulator  $\text{SmB}_6$  measured in [19, 20], where  $T^*$  is the onset temperature of the surface resistivity plateau,  $T_{FL}$  marks the onset of Fermi liquid behavior in the high pressure metallic phase.  $\Delta_1$  and  $\Delta_2$  are the transport and optical gaps, respectively. (b) Sketch of the 5d neutron form-factor seen in the  $\text{SmB}_6$  magnetic exciton from [21]. Mysteriously, even though the exciton involves hybridized f-electrons, the form factor is that of 5d electrons, as if the f-electrons have a negligible g-factor.

where an insulator-metal transition occurs. Sun's group finds that the indirect transport gap  $\Delta_1$  of the insulator collapses to zero at the insulator-metal transition, but the larger hybridization (direct) gap  $\Delta_2$  remains finite well into the metallic regime. The characteristic Fermi scale of the metal appears to grow linearly with pressure beyond the insulator-metal transition.

- Colin Broholm showed his groups measurement of a magnetic exciton as the gap develops in  $\text{SmB}_6$  [21–23]. This is another indication of the close vicinity of magnetic order. A collective, dispersing magnetic exciton in  $\text{SmB}_6$  was first observed by Alekseev and collaborators back in the 1990s [22, 23], but I think its fair to say that it is only recently, that its connection with a hybridized Kondo insulator model has been understood in terms of a theoretical model [21, 24]. This data can be fit to a theory of an ingap magnetic exciton formation between hybridized d- and f-electrons [21, 24]. Yet mysteriously, in the neutron data reveals a d-electron form factor, as if the vital f-electron part of the exciton is entirely invisible to neutrons (Fig. 5b) *Is this because the f-electron's have a very low g-factor, or is it an indication of something we have overlooked in the bulk physics?*
- Suchitra Sebastian and Lu Li presented the results of dHvA measurements on this system. Lu Li argued that the angular dependence of his data imply two dimensional, topological surface states consistent with the idea that  $\text{SmB}_6$  is a strong topological insulator. This alone is exciting. By contrast, Suchitra Sebastian argued that her group's results suggest a gapless, albeit insulating bulk with 3 dimensional Fermi surfaces. In support of this idea, Sebastian argued that the strong dependence of the effective mass in the orbits at low temperatures is consistent with the large bulk linear specific heat  $C_V \sim \gamma T$  that has been long-observed in this material and presented thermal conductivity data that indicate that the thermal conductivity over temperature acquires a finite zero temperature intercept  $\kappa(T, H)/T \neq 0|_{T \rightarrow 0}$  in a magnetic field.

These new developments suggest that while  $\text{SmB}_6$  does have robust topological



surface states, there is something unusual about the sub-gap bulk electrodynamics and magnetism of its f-electrons. Is it possible that the close vicinity to magnetism transforms this material from a conventional band-gapped topological insulator, into a *strange insulator*, with a Fermi surface of gapless excitations that are responsible for the linear specific heat, the thermal conductivity and dHvA oscillations? Were this true, it would pose a most fascinating paradox: for how can quasiparticles exhibit Landau quantization - a semi-classical signature of the Lorentz force, while remaining an insulator? My own view on this material is in a state of evolution. A year ago Erten, Ghaemi and I argued that the observed properties of  $\text{SmB}_6$  are consistent with topological surface states, combined with a break-down of the Kondo effect at the surface[25] but the latest data, with a far more detailed study of the angular dependence of the low frequency modes, and the unusual thermal conductivity suggest that this view needs re-evaluation. The new results motivate considering the possibility of a new kind of insulator, one in which a neutral shadow of the original Fermi surface remains unhybridized, yet insulating. Could this be a gapless spinon Fermi surface, perhaps a  $Z_2$  spin liquid lurking within the material? Alternatively, the idea of a Majorana Fermi surface, proposed by Eduardo Miranda, Alexei Tsvelik and myself many years ago and renovated by Ganputhy Baskaran may prove useful[26, 27]. According to this radical view,  $\text{SmB}_6$  would be a failed superconductor with neutral current excitations that can link selectively to the Lorentz force, thanks to an almost broken  $U(1)$  gauge symmetry[28].

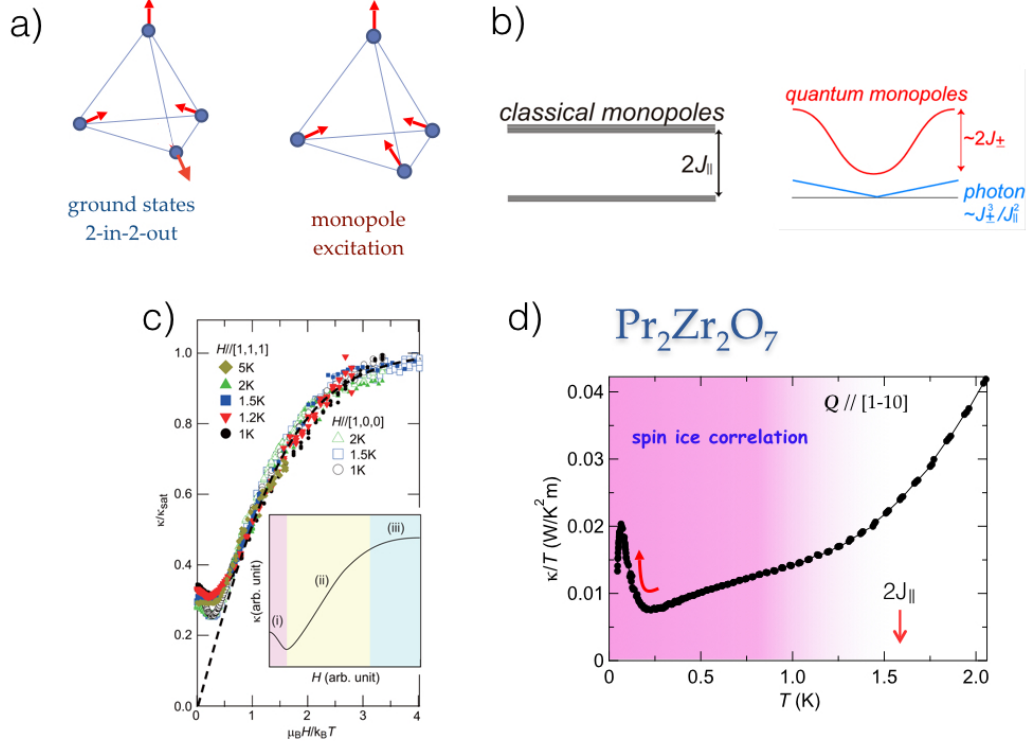
I mention in passing that there may be many other heavy fermion systems with topological surface states. We didn't hear much about these possibilities at this meeting, but plateau conductivities are present in old data on Sb doped  $\text{CeNiSn}$ [29] and  $\text{Ce}_3\text{Pt}_4\text{Bi}_3$  under pressure[30], which may also be topological Kondo insulators, protected by their non-symorphic crystal symmetries[31]

## 5. Quantum Spin Ice

Frustrated spin systems offer the opportunity to observed strongly correlated behavior in quantum materials without the complications of charge motion. One of the areas of particular interest in recent years has been that of “spin ice” pyrochlore magnets. These uniquely frustrated magnets contain spins arranged on the vertices of tetrahedra arranged in pyrochlore lattice, which obey the “two-in, two-out” ice rules. Excitations out of the spin-ice manifold form magnetic monopoles with an excitation energy of  $2J_{\parallel}$  (Fig. 6 a & b). This has raised the fascinating possibility that when quantum fluctuations are included, spin ice will melt, to form a quantum spin liquid. Such quantum spin ice is predicted to have itinerant monopoles and an “photon” mode associated with its emergent electric and magnetic fields[34].

At this meeting I was particularly fascinated by new results on two spin-ice pyrochlore lattices  $\text{Yb}_2\text{Ti}_2\text{O}_7$  and  $\text{Pr}_2\text{Zr}_2\text{O}_7$  by Yoshi Tokiwa and collaborators [33, 35].  $\text{Yb}_2\text{Ti}_2\text{O}_7$  ferromagnetically orders below  $T_C = 0.2K$ , but above this temperature, based on their measurements, Tokiwa et al. propose it is pyrochlore quantum spin liquid with itinerant quantum monopoles. By contrast,  $\text{Pr}_2\text{Zr}_2\text{O}_7$  does not magnetically order down to the lowest temperatures.

In  $\text{Yb}_2\text{Ti}_2\text{O}_7$ [33] Tokiwa et al have measured the thermal conductivity  $\kappa(T, H)$  versus field  $H$  and temperature  $T$ . At high temperatures, the thermal conductivity is a function of  $\mu_B H/T$ , and can be understood as a result of spin-phonon scattering. However, at low fields, the thermal conductivity  $\kappa(H)/T$  decreases with



**Figure 6.** a) In “spin ice” the ground-state of each tetrahedron satisfies the ice rules (two spins in, two spins out)[32]. “Monopole” excitations correspond to three out/in, one in/out. b) Classical monopoles are localized and have an excitation energy of  $2J_{\parallel}$ . Quantum monopoles are itinerant, and theory also predicts the development of emergent “photon” excitations. c) Field dependence of thermal conductivity in  $\text{Yb}_2\text{Ti}_2\text{O}_7$  from [33] at various temperatures. d) Temperature dependence of thermal conductivity  $\kappa/T$  in spin ice compound  $\text{Pr}_2\text{Zr}_2\text{O}_7$  reported by Tokiwa et al., showing marked upturn below 0.2K which is interpreted as a signal of the spin-photon.

applied field, which the authors interpret as a result of Monopole thermal conductivity (6 c ). This is a very interesting result, and it will be interesting to examine whether at still lower temperatures, a thermal conductivity from the spin-photons can be observed.

In the sister compound  $\text{Pr}_2\text{Zr}_2\text{O}_7$ [35], Tokiwa reports that the system displays no discernable long range order, and that the measured inelastic neutron scattering dominates 90% of the scattered intensity, consistent with a quantum spin ice compound. Perhaps most excitedly, the thermal conductivity  $\kappa/T$  displays a marked upturn at low temperatures (Fig. 6d) that may be the first signs of the fabled spin-photon excitation. There is a hope that future neutron measurements will be able to resolve and confirm the presence of this excitation. This is a developing story to keep your eyes on.

## 6. Iron Based Superconductors

Unlike the cuprate superconductors, where we have unambiguous evidence for the symmetry and structure of the pair wavefunction, in the iron based superconductors, this issue is still a matter of continuing discussion and fascination[36]. Of particular

interest, is the mechanism by which the pair condensate overcomes the considerable Coulomb repulsion between electrons on the iron sites.

Most iron based superconductors are fully gapped, and the canonical theory for their gap structure supposes that it has  $s^\pm$  structure, with an s-symmetry. In the  $s^\pm$  scenario, spin fluctuations exchanged between the electron and hole bands lead to a sign difference between the electron and hole pockets. This sign difference gives rise to a suppression of the local s-wave pair density, and thereby overcomes the strong onsite Coulomb interactions.

The discovery of a number of “bad-actor” iron-based superconductors which only have electron, or hole bands, poses a difficulty to the  $s^\pm$  pairing mechanism. One interesting idea in this respect, is the possibility that the orbital quantum numbers of the paired quasiparticles can entangle with the pair condensate, in a fashion reminiscent of the spin entanglement present in triplet superfluids or superconductors[37–40]. In an conventional scenario the pair condensate wavefunction is diagonal in the orbital indices

$$\langle d_{\mu\uparrow}(\mathbf{k})d_{\nu\downarrow} \rangle \propto \delta_{\mu\nu}\Delta(\mathbf{k}) \quad (6)$$

whereas in an orbitally entangled scenario, the pair wavefunction has a non-trivial orbital dependence

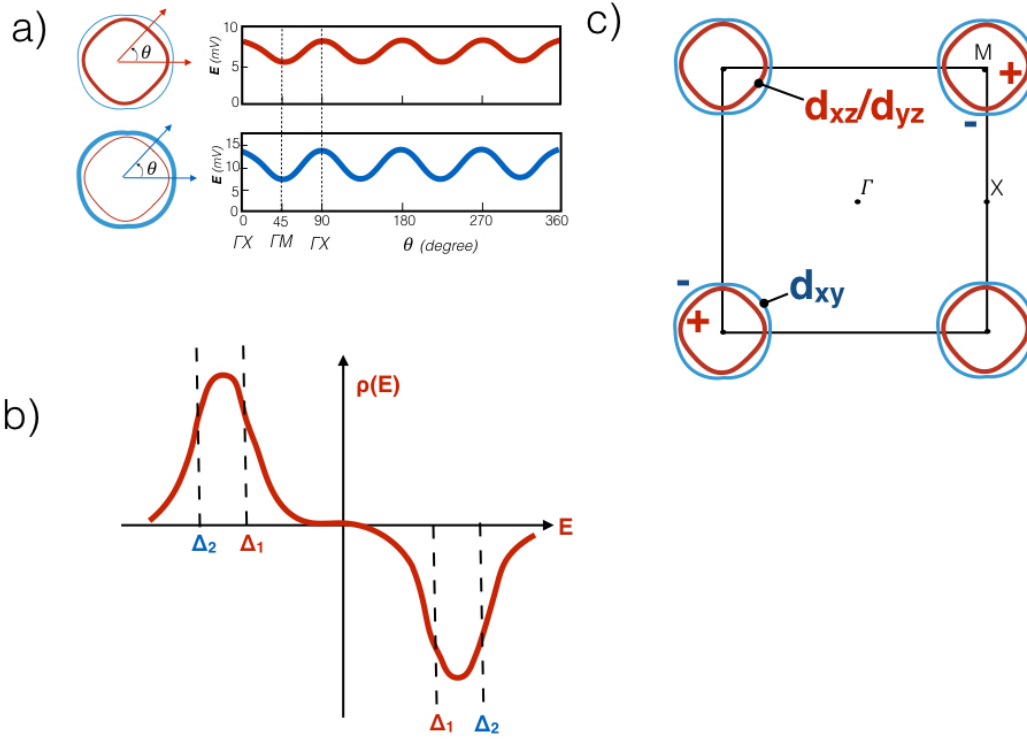
$$\langle d_{\mu\uparrow}(\mathbf{k})d_{\nu\downarrow} \rangle \propto \sum_{\Gamma} \Delta_{\Gamma}(\mathbf{k})\alpha_{\mu\nu}^{\Gamma}, \quad (7)$$

where the  $\alpha^{\Gamma}$  are matrices in orbital space and the functions  $\Delta_{\Gamma}$  are their corresponding gap functions.

One of the iron-based superconductors with exclusively electron pockets, is the single layer FeSe iron chalcogenide, with a transition temperature  $T_c$  in excess of 60K, yet which only has electron pockets centered around the  $M$  points. At this meeting, the group of Hai Hu Wen announced a new set of scanning tunneling spectroscopy measurements on the layered iron chalcogenide  $(\text{Li}_{1-x}\text{Fe}_x)\text{OHFeSe}$ , a bulk analogue of FeSe with a  $T_c$  of 40K that appears to confirm an orbital dependence to the pair wavefunction. These measurements (Fig. 7 a) confirm that the electron pockets have a fully developed gap[41, 42]. The large ratio  $2\Delta/T_c \sim 9$  indicates that the pairing is in the strong coupling limit. Wen’s group employ some recent theoretical analyses of the effects of pairing on interband quasiparticle scattering[43] to show that gaps on the two hole pockets are in *antiphase*. Here the basic idea is that interband scattering between two Fermi surfaces with opposite pairing signs gives rise to a correction to the density of states that is antisymmetric in energy and peaked at energies  $E \in [|\Delta_1|, |\Delta_2|]$ . It is this feature that is observed in the data (Fig. 7 b). This is particularly interesting, for the outermost electron pocket has a predominantly  $d_{xy}$  character, while the innermost hole pocket have  $d_{zx}/d_{zy}$  character. These results seem to be consistent with orbital antiphase pairing.

These new results will doubtless stimulate further work in this direction. Some of the questions that they pose are:

- Is there a non-trivial dependence of the pairing on the other  $d_{zx}$  and  $d_{zy}$  orbitals?
- Does the strong-coupling limit of the pairing in the iron-based superconductors have a local interpretation, e.g in terms of localized bosons or perhaps composites of pairs and spins?



**Figure 7.** Sketch of STM Data taken on  $(\text{Li}_{1-x}\text{Fe}_x)\text{OHFeSe}$  from [41] a) showing two fully gapped electron pockets around  $M$  points b) sketch of antisymmetric contribution to the quasiparticle-interference spectrum, peaked between energies  $|\Delta_1|$  and  $|\Delta_2|$ , a signature of potential scattering between two Fermi surfaces with opposite gap sign c) inferred antiphase structure of the gaps on the hole pockets.

## 7. Strange Metals and Quantum Criticality

Historically, quantum criticality in strongly correlated, particularly heavy fermion systems has been cast in terms of the Doniach competition between the RKKY and the Kondo interactions. Over the past decade or so, our community has become increasingly aware of a second quantum criticality axis - that of quantum and geometric frustration, which can also drive local moment systems from ordered magnetism into paramagnetic spin liquids or valence bond solids. The combined influence of these two effects is notionally mapped out in the global phase diagram[44–47] for the Kondo lattice (Fig. 8).

SCES 2016 was marked by the entry of several new materials that illustrate different aspects of the global phase diagram for quantum criticality. In particular:

- (1) **1 D spin liquid behavior in a heavy fermion compound.** The observation of spinons in the quasi-one dimensional heavy fermion magnet  $\text{Yb}_2\text{Pt}_2\text{Pb}$  [48]. This material illustrates the feasibility of spin-liquid formation within a strongly spin-orbit coupled heavy fermion material (See: Fig. 8 a ).
- (2) **Frustration tuned non-Fermi liquid behavior** was observed in the heavy fermion Kagome-lattice compound  $\text{CeRhSn}$  [49]. Here, careful Grüneisen parameter measurements reveal that the application of strain which selectively relieves the Kagome-lattice frustration, drives the strange metal state back into a Fermi liquid (Fig. 8 b).
- (3) **Strange metals, robust against pressure.** At this meeting, two strange metallic Yb heavy fermion compounds, the hexagonal layered system

$\beta$ -YbAlB<sub>4</sub>[50] and the heavy fermion quasi-crystal YbAlAu[51] were discussed. Both of these systems exhibit field-tune quantum criticality - namely power-law behavior in their specific heat and magnetic susceptibility, which is field tuned and reverts to Fermi liquid behavior in the smallest applied magnetic fields. Remarkably, the application of pressure fails to remove this quantum critical behavior in either compound, suggesting the formation of a strange metallic phase. Also, Deguchi finds that strange metallic behavior is only present in pristine quasicrystals of YbAlAu[51] while the approximant crystal Au<sub>51</sub>Al<sub>35</sub>Yb<sub>14</sub>, with precisely the same local structure, but a finite unit cell size, is a Fermi liquid (Fig. 8 c).

This is clearly a gold-mine for future theoretical activity. I would like to mention a few key questions and observations taken away from these results

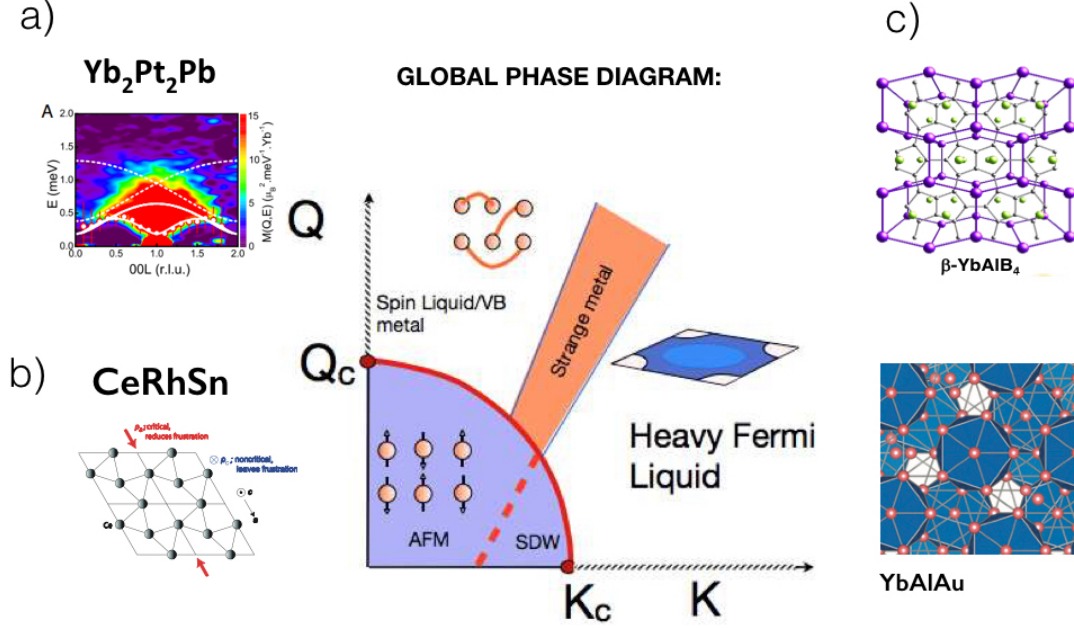
- CeRhSn,  $\beta$ -YbAlB<sub>4</sub>[50] and YbAlAu[51] are each examples of metals in which the non-Fermi liquid behavior persists over a range of parameters. From these materials I think it is clear that we should take the idea of strange metal phases seriously. What is the theoretical basis for such phases?
- What is the role of valence fluctuations? Kazumasa Miyake and Shinji Watanabe[52, 53] argue that quantum critical valence fluctuations are the driver for strange metallic behavior. Certainly, the strange metallic behavior in  $\beta$ -YbAlB<sub>4</sub>[50] and YbAlAu is clearly separated from magnetic quantum criticality. We need to understand if valence fluctuations can account for strange metallic phases, or whether alternative mechanisms are needed.
- Why is the pristine quasicrystal YbAlAu a strange metal, yet its approximant Au<sub>51</sub>Al<sub>35</sub>Yb<sub>14</sub>[51] a Fermi liquid? Since there is no local difference in the electronic structure or chemistry of these two compounds, we might speculate that the quasicrystal must contain some kind of electronic or magnetic criticality that is cut-off by the finite size of the unit cell in the approximant.
- We heard from Yifeng Yang[54] about the two-fluid phenomenology of heavy fermion materials used to understand heavy fermion systems on the brink of magnetism. One of the most striking examples of such behavior is CeRhIn<sub>5</sub> in which the Cerium f-moments homogeneously time-share between local moment magnetism and superconductivity. I think the time has come to think of ways to try to cast these ideas into the language of a wavefunction and I'd like to encourage the community ways of doing this. My own favorite, formulated by my former student Aline Ramires, is the possibility of a Gutzwiller projected product of a bosonic magnetic fluid and a fermionic Kondo liquid

$$|\Psi\rangle = P_G(|\Psi_F\rangle \otimes |\Psi_B\rangle),$$

where  $P_G$  is a Gutzwiller projection that entangles the product of a fermionic  $|\Psi_F\rangle$  and bosonic  $|\Psi_B\rangle$  representation of the spin background[55]. My main point however, is that the theory community needs to be thinking about new approaches and next generation wavefunctions that may help us to understand the entangled nature of magnetism and superconductivity.

## Acknowledgments

This research was supported by the United States Department of Energy Basic Energy Sciences grant DE-FG02-99ER45790 and United States National Science Foundation grant NSF DMR 1309929. This article was written while at the Aspen Center



**Figure 8.** Global Phase diagram for heavy fermions [44–47] with horizontal “Kondo” and quantum frustration (Q) axes. a) Spinon excitations in the inelastic neutron scattering spectrum of  $\text{Yb}_2\text{Pt}_2\text{Pb}$ , figure [48, 56] b)  $\text{CeRhSn}$  in which only strain that relieves the frustration is a relevant perturbation to the non-Fermi liquid ground-state [49] c) the layered crystal  $\beta\text{-YbAlB}_4$  [50] (Figure courtesy of Yosuke Matsumoto) and the quasicrystal  $\text{YbAlAu}$  [51] (Figure courtesy of Kazuhiko Deguchi) both exhibit field-tuned quantum criticality, disconnected from magnetic criticality and robust against pressure.

for Physics, which is supported by National Science Foundation grant PHY-1066293. I should like to thank Meigan Aronson, Colin Broholm, Girsh Blumberg, Po-Yao Chang, Xi Dai, Onur Erten, Gilbert Lonzarich, Suchitra Sebastian, Liling Sun, Joe Thompson, Yoshi Tokiwa, Hai Hu Wen and Changming Yue for discussions related to their work and this talk. My thanks to the organizers of SCES 2016, which was truly a memorable conference. Finally, my sincere apologies to the authors of the many wonderful talks and posters that I heard and visited, but was unable to include in this perspective. Your work is great and I look forward to learning how it now develops.

## References

- [1] P. Coleman, *Many Body Physics: Unfinished Revolution*, Annales Henri Poincaré 4 (2003), pp. 559–580, Available at <http://dx.doi.org/10.1007/s00023-003-0943-9>.
- [2] S. Hartnoll, *Lectures on holographic methods for condensed matter physics*, arXiv:0903.3246 (2009), Available at <http://arxiv.org/abs/0903.3246>.
- [3] C. Domb, *Critical phenomena: A brief historical survey*, Contemporary Physics 26 (1985), pp. 49–72, Available at <http://www.tandfonline.com/doi/abs/10.1080/00107518508210738?journalCode=tcph20>.
- [4] Z.X. Li, Y.F. Jiang, and H. Yao, *Solving the fermion sign problem in quantum Monte Carlo simulations by Majorana representation*, Phys. Rev. B 91 (2015), p. 241117, Available at <http://link.aps.org/doi/10.1103/PhysRevB.91.241117>.
- [5] Z.X. Li, Y.F. Jiang, and H. Yao, *Majorana-time-reversal symmetries: a fundamental principle for sign-problem-free quantum Monte Carlo simulations*, arXiv:1601.05780 (2016), Available at <http://arxiv.org/abs/1601.05780>.
- [6] C. Yue, Y. Wang, and X. Dai, *An Efficient Continuous-time Quantum Monte-Carlo Impu-*



- ity Solver in Kondo Regime, Proceedings of SCES 2016, Journal of Physics: Conference Series (2016).
- [7] J.R. Schrieffer and P. Wolff, *Relation between the Anderson and Kondo Hamiltonians*, Phys. Rev. 149 (1966), p. 491, Available at <http://link.aps.org/doi/10.1103/PhysRev.149.491>.
  - [8] M.R. Norman, D. Pines, and C. Kallin, *The pseudogap: friend or foe of high  $T_c$ ?*, Advances in Physics 54 (2005), pp. 715–753, Available at <http://www.tandfonline.com/doi/abs/10.1080/00018730500459906>.
  - [9] J.A. Mydosh and P.M. Oppeneer, *Colloquium: Hidden order, superconductivity, and magnetism: The unsolved case of  $URu_2Si_2$* , Reviews of Modern Physics 83 (2011), pp. 1301–1322, Available at <http://link.aps.org/doi/10.1103/RevModPhys.83.1301>.
  - [10] I.M. Vishik, M. Hashimoto, R.H. He, W.S. Lee, F. Schmitt, D. Lu, R.G. Moore, C. Zhang, W. Meevasana, T. Sasagawa, S. Uchida, K. Fujita, S. Ishida, M. Ishikado, Y. Yoshida, H. Eisaki, Z. Hussain, T.P. Devereaux, and Z.X. Shen, *Phase competition in trisected superconducting dome*, Proceedings of the National Academy of Sciences 109 (2012), pp. 18332–18337, Available at <http://www.pnas.org/content/109/45/18332.full>.
  - [11] M. Le Tacon, A. Bosak, S.M. Souliou, G. Dellea, T. Loew, R. Heid, K.P. Bohnen, G. Ghiringhelli, M. Krisch, and B. Keimer, *Inelastic X-ray scattering in  $YBa_2Cu_3O_{6.6}$  reveals giant phonon anomalies and elastic central peak due to charge-density-wave formation*, Nature Physics 10 (2013), pp. 52–58, Available at <http://www.nature.com/doifinder/10.1038/nphys2805>.
  - [12] Y.H. Liu, R.M. Konik, T.M. Rice, and F.C. Zhang, *Giant phonon anomaly associated with superconducting fluctuations in the pseudogap phase of cuprates*, Nature Communications 7 (2016), Available at <http://www.nature.com/ncomms/2016/160120/ncomms10378/full/ncomms10378.html>.
  - [13] A. Dubroka, M. Rössle, K.W. Kim, V.K. Malik, D. Munzar, D.N. Basov, A.A. Schafgans, S.J. Moon, C.T. Lin, D. Haug, V. Hinkov, B. Keimer, T. Wolf, J.G. Storey, J.L. Tallon, and C. Bernhard, *Evidence of a Precursor Superconducting Phase at Temperatures as High as 180 K in  $RBa_2Cu_3O_{7-\delta}$  ( $R = Y, Gd, Eu$ ) Superconducting Crystals from Infrared Spectroscopy*, Phys. Rev. Lett. 106 (2011), p. 047006, Available at <http://link.aps.org/doi/10.1103/PhysRevLett.106.047006>.
  - [14] H.H. Kung, R. Baumbach, E. Bauer, V.K. Thorsmolle, W.L. Zhang, K. Haule, J.A. Mydosh, and G. Blumberg, *Chirality density wave of the “hidden order” phase in  $URu_2Si_2$* , Science (New York, NY) 347 (2015), pp. 1339–1342, Available at <http://www.sciencemag.org/content/347/6228/1339.full>.
  - [15] H.H. Kung, R. Baumbach, E. Bauer, S. Ran, N. Kanchanavatee, J. Mydosh, K. Haule, B. Maple, and G. Blumberg, *The analogy between the ‘hidden order’ and the orbital antiferromagnetism in  $URu_2Si_2$* , arXiv.org (2016), Available at <http://arxiv.org/abs/1608.01748v1>.
  - [16] P. Chandra, P. Coleman, and M. Continentino, to be published (2016).
  - [17] P. Coleman and A.J. Schofield, *How should we interpret the two transport relaxation times in the cuprates?*, Journal of Physics Condens. Matt. 8 (1996), pp. 9985–10015, Available at <http://iopscience.iop.org/article/10.1088/0953-8984/8/48/020>.
  - [18] M. Dzero, J. Xia, V. Galitski, and P. Coleman, *Topological Kondo Insulators*, Annual Review of Condensed Matter Physics 7 (2016), pp. 249–280, Available at <http://www.annualreviews.org/doi/abs/10.1146/annurev-conmatphys-031214-014749>.
  - [19] Y. Zhou, Q. Wu, P.F.S. Rosa, R. Yu, J. Guo, W. Yi, S. Zhang, Z. Wang, H. Wang, S. Cai, K. Yang, A. Li, Z. Jiang, S. Zhang, X. Wei, Y. Huang, Y.f. Yang, Z. Fisk, Q. Si, L. Sun, and Z. Zhao, *Quantum phase transition and destruction of Kondo effect in pressurized  $SrB_6$* , arXiv:1603.05607 (2016), Available at <http://arxiv.org/abs/1605.00416>.
  - [20] L. Sun and Q. Wu, *Pressure-induced exotic states in rare earth hexaborides*, Reports on Progress in Physics 79 (2016), p. 084503, Available at <http://iopscience.iop.org/article/10.1088/0034-4885/79/8/084503>.
  - [21] W.T. Fuhrman, J. Leiner, P. Nikolić, G.E. Granroth, M.B. Stone, M.D. Lumsden, L.



- DeBeer-Schmitt, P.A. Alekseev, J.M. Mignot, S.M. Koohpayeh, P. Cottingham, W.A. Phelan, L. Schoop, T.M. McQueen, and C. Broholm, *Interaction Driven Subgap Spin Exciton in the Kondo Insulator  $\text{SmB}_6$* , Physical Review Letters 114 (2015), p. 036401, Available at <http://link.aps.org/doi/10.1103/PhysRevLett.114.036401>.
- [22] P.A. Alekseev, J.M. Mignot, J. Rossat-Mignod, V.N. Lazukov, I.P. Sadikov, E.S. Konovalova, and Y.B. Paderno, *Magnetic excitations spectrum in  $\text{SmB}_6$  single-crystals*, Physica-B 186-88 (1993), pp. 384–386, Available at <http://www.sciencedirect.com/science/article/pii/092145269390580Y>.
- [23] P.A. Alekseev, J.M. Mignot, J. Rossat-Mignod, V.N. Lazukov, I.P. Sadikov, E.S. Konovalova, and Y.B. Paderno, *Magnetic excitation spectrum of mixed-valence  $\text{SmB}_6$  studied by neutron scattering on a single crystal*, J. Phys. Condens. Matter 7 (1995), p. 289, Available at <http://iopscience.iop.org/article/10.1088/0953-8984/7/2/007>.
- [24] W.T. Fuhrman and P. Nikolić, *In-gap collective mode spectrum of the topological Kondo insulator  $\text{SmB}_6$* , Phys. Rev. B 90 (2014), Available at <http://arxiv.org/abs/1409.3220v1>.
- [25] O. Erten, P. Ghaemi, and P. Coleman, *Kondo Breakdown and Quantum Oscillations in  $\text{SmB}_6$* , Physical review letters 116 (2016), pp. 046403–5, Available at <http://link.aps.org/doi/10.1103/PhysRevLett.116.046403>.
- [26] A. Tsvelik, P. Coleman, and E. Miranda, *Are Kondo insulators gapless?*, Physica B-Condensed Matter 186-188 (1993), pp. 362–364, Available at <http://www.sciencedirect.com/science/article/pii/092145269390574P>.
- [27] G. Baskaran, *Majorana Fermi Sea in Insulating  $\text{SmB}_6$ : A proposal and a Theory of Quantum Oscillations in Kondo Insulators*, arXiv.org (2015), Available at <http://arxiv.org/abs/1507.03477v1>.
- [28] O. Erten, P.Y. Chang, and P. Coleman, work in progress (2016).
- [29] A. Ślebarski and J. Spalek, *From Kondo semimetal to spin-glass behaviour in doped  $\text{CeNi}_{1-\delta}\text{Sn}_{1+\delta-x}\text{Sb}_x$* , Philosophical Magazine 89 (2009), pp. 1845–1859, Available at <http://dx.doi.org/10.1080/14786430802647073>.
- [30] J.C. Cooley, M.C. Aronson, and P.C. Canfield, *High pressures and the Kondo gap in  $\text{Ce}_3\text{Bi}_4\text{Pt}_3$* , Phys. Rev. B 55 (1997), pp. 7533–7538, Available at <http://link.aps.org/doi/10.1103/PhysRevB.55.7533>.
- [31] P.Y. Chang, O. Erten, and P. Coleman, *Möbius Kondo Insulators*, arXiv:1603.03435 (2016), Available at <http://arxiv.org/abs/1603.03435>.
- [32] M.J.P. Gingras and P.A. McClarty, *Quantum spin ice: a search for gapless quantum spin liquids in pyrochlore magnets*, Reports on Progress in Physics 77 (2014), p. 056501, Available at <http://stacks.iop.org/0034-4885/77/i=5/a=056501>.
- [33] Y. Tokiwa, T. Yamashita, M. Udagawa, S. Kittaka, T. Sakakibara, D. Terazawa, Y. Shimoyama, T. Terashima, Y. Yasui, T. Shibauchi, and Y. Matsuda, *Possible observation of highly itinerant quantum magnetic monopoles in the frustrated pyrochlore  $\text{Yb}_2\text{Ti}_2\text{O}_7$* , Nature Communications 7 (2016), Available at <http://www.nature.com/ncomms/2016/160225/ncomms10807/full/ncomms10807.html>.
- [34] M. Hermele, M.P.A. Fisher, and L. Balents, *Pyrochlore photons: The  $u(1)$  spin liquid in a  $s = \frac{1}{2}$  three-dimensional frustrated magnet*, Phys. Rev. B 69 (2004), p. 064404, Available at <http://link.aps.org/doi/10.1103/PhysRevB.69.064404>.
- [35] K. Kimura, S. Nakatsuji, J.J. Wen, C. Broholm, M.B. Stone, E. Nishibori, and H. Sawa, *Quantum fluctuations in spin-ice-like  $\text{Pr}_2\text{Zr}_2\text{O}_7$* , Nature Communications 4 (1), pp. 1–6, Available at [NatureCommunications4, \(2013\).doi:10.1038/ncomms2914](http://www.nature.com/naturecommunications/2013/doi:10.1038/ncomms2914).
- [36] Q. Si, R. Yu, and E. Abrahams, *High-temperature superconductivity in iron pnictides and chalcogenides*, Nature Reviews Materials 1 (2016), pp. 16017–15, Available at <http://www.nature.com/articles/natrevmats201617>.
- [37] Z.P. Yin, K. Haule, and G. Kotliar, *Spin dynamics and orbital-antiphase pairing symmetry in iron-based superconductors*, Nature Physics 10 (2014), pp. 845–850, Available at <http://www.nature.com/doi/10.1038/nphys3116>.
- [38] R. Yu, J.X. Zhu, and Q. Si, *Orbital-selective superconductivity, gap anisotropy, and spin*

- resonance excitations in a multiorbital  $t$ - $J_1$ - $J_2$  model for iron pnictides, Phys. Rev. B 89 (2014), p. 024509, Available at <http://link.aps.org/doi/10.1103/PhysRevB.89.024509>.
- [39] T. Ong, P. Coleman, and J. Schmalian, *Concealed d-wave pairs in the  $s^\pm$  condensate of iron-based superconductors*, Proceedings of the National Academy of Sciences 113 (2016), pp. 5486–5491, Available at <http://www.pnas.org/content/113/20/5486.full>.
  - [40] R. Nourafkan, G. Kotliar, and A.M.S. Tremblay, *Correlation-enhanced odd-parity inter-orbital singlet pairing in the iron-pnictide superconductor LiFeAs*, arXiv.org (2015), Available at <http://arxiv.org/abs/1508.01789v2>.
  - [41] Z. Du, X. Yang, D. Fang, G. Du, J. Xing, H. Yang, X. Zhu, and H.H. Wen, To be published (2016).
  - [42] Z. Du, X. Yang, D. Fang, G. Du, J. Xing, H. Yang, X. Zhu, and H.H. Wen, *Scrutinizing the double superconducting gaps and strong coupling pairing in  $(\text{Li}_{1-x}\text{Fe}_x)\text{OFeSe}$* , Nature Communications 7 (2016), p. 10565, Available at <http://www.nature.com/ncomms/2016/160129/ncomms10565/full/ncomms10565.html>.
  - [43] P.J. Hirschfeld, D. Altenfeld, I. Eremin, and I.I. Mazin, *Robust determination of the superconducting gap sign structure via quasiparticle interference*, Phys. Rev. B 92 (2015), p. 184513, Available at <http://link.aps.org/doi/10.1103/PhysRevB.92.184513>.
  - [44] T. Senthil, M. Vojta, and S. Sachdev, *Weak magnetism and non-Fermi liquids near heavy-fermion critical points*, Phys. Rev. B 69 (2004), p. 035111, Available at <http://link.aps.org/doi/10.1103/PhysRevB.69.035111>.
  - [45] Q. Si, *Global magnetic phase diagram and local quantum criticality in heavy fermion metals*, Physica B-Condensed Matter 378-380 (2006), pp. 23–27, Available at <http://linkinghub.elsevier.com/retrieve/pii/S092145260600007X>.
  - [46] E. Lebanon and P. Coleman, *Fermi liquid identities for the infinite-  $U$  multichannel Anderson model*, Phys. Rev. B 76 (2007), p. 085117, Available at <http://link.aps.org/doi/10.1103/PhysRevB.76.085117>.
  - [47] P. Coleman and A.H. Nevidomskyy, *Frustration and the Kondo effect in heavy fermion materials*, Journal of Low Temperature Physics (2010), Available at <http://link.springer.com/article/10.1007/s10909-010-0213-4>.
  - [48] L.S. Wu, W.J. Gannon, I.A. Zaliznyak, A. Tsvelik, M. Brockmann, J.S. Caux, M.S. Kim, Y. Qiu, J.R.D. Copley, G. Ehlers, A. Podlesnyak, and M. Aronson, *Orbital-exchange and fractional quantum number excitations in an f-electron metal,  $\text{Yb}_2\text{Pt}_2\text{Pb}$* , Science (New York, NY) 352 (2016), pp. 1206–1210, Available at <http://science.sciencemag.org/content/352/6290/1206.abstract>.
  - [49] Y. Tokiwa, C. Stingl, M.S. Kim, T. Takabatake, and P. Gegenwart, *Characteristic signatures of quantum criticality driven by geometrical frustration*, Science Advances 1 (2015), pp. e1500001–e1500001, Available at <http://advances.sciencemag.org/cgi/doi/10.1126/sciadv.1500001>.
  - [50] T. Tomita, K. Kuga, Y. Uwatoko, P. Coleman, and S. Nakatsuji, *Strange metal without magnetic criticality*, Science (New York, NY) 349 (2015), pp. 506–509, Available at <http://science.sciencemag.org/content/349/6247/506.abstract>.
  - [51] K. Deguchi, S. Matsukawa, N.K. Sato, T. Hattori, K. Ishida, H. Takakura, and T. Ishimasa, *Quantum critical state in a magnetic quasicrystal*, NATURE MATERIALS 11 (2012), pp. 1–4, Available at <http://dx.doi.org/10.1038/nmat3432>.
  - [52] S. Watanabe and K. Miyake, *Roles of critical valence fluctuations in Ce- and Yb-based heavy fermion metals*, J. Phys. Cond. Matt. 23 (2011), p. 094217.
  - [53] K. Miyake and S. Watanabe, *Ubiquity of Unconventional Quantum Criticality due to Critical Valence Fluctuations in Heavy Fermion Metals*, Proceedings of SCES 2016, Philosophical Magazine (2016).
  - [54] Y.f. Yang, Z. Fisk, H.O. Lee, J.D. Thompson, and D. Pines, *Scaling the Kondo lattice*, Nature 454 (2008), pp. 611–613.
  - [55] A. Ramires and P. Coleman, *Supersymmetric approach to heavy fermion systems*, Physical Review B 93 (2016), p. 035120, Available at <http://journals.aps.org/prb/abstract/>

[10.1103/PhysRevB.93.035120](#).

- [56] L.S. Wu, W.J. Gannon, I.A. Zaliznyak, A. Tsvelik, M. Brockmann, J.S. Caux, M.S. Kim, Y. Qiu, J.R.D. Copley, G. Ehlers, A. Podlesnyak, and M. Aronson, *Orbital-exchange and fractional quantum number excitations in an f-electron metal,  $\text{Yb}_2\text{Pt}_2\text{Pb}$* , (Figure from arXiv:1606.01309 with permission from authors.) .